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A Generalized Slutsky Matrix of the Second Kind

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1. INTRODUCTION

Here we consider a parameterized optimization problem. Let the parameter spaces A, B be open sets in R^p, R^m, respectively. Denote by $f(\cdot, a)$, $g(\cdot, a) = {}^t(g_1(\cdot, a), ..., g_m(\cdot, a))$ the object function and the vector constraints defined over an open set X in $R^n (n \geqslant m)$. Assume that, to each (a, b) in $A \times B$, $x(a, b)$ maximizes (locally) $f(x, a)$ subject to the vector constraint $g(x, a) = b$ with a vector Lagrangian multiplier $\lambda(a, b)$ in R^m. The central problems examined here are the problem (A) and its converse problem (B).

PROBLEM (A). Study the basic properties of the functions $x(a, b)$ and $\lambda(a, b)$, as (a, b) varying in $A \times B$.

PROBLEM (B). Given constraint g, a function x (and λ) satisfying the basic properties found in Problem (A), can x (and λ) be gnerated by the maximization of an object function f subject to $g = b$?

These problems are motivated from mathematical economics. Let the object function be a utility function $U(x)$, and the constraint be a budget equation $p \cdot x = I$, where x, p, I stands for commodities, prices, and an income. Thus $x(p, I)$ is the demand function derived from the utility function $U(x)$. In this case, Problems (A), (B) have a very satisfactory answer. The substitution matrix $S_{kj} = \partial x_k / \partial p_j + x_j(\partial x_k / \partial I)$, defined by Slutsky [7] is shown to be symmetric (Slutsky), negative semidefinite (Johnson, Hicks, and Allen). Furthermore, one can recover $U(x)$ from $x(p, I)$ if it has a symmetric, negative semidefinite substitution matrix (Samuelson). The matrix S_{kj} can also be defined, in a straightforward manner, for a general parametrized problem (see [1] and Definition 2). Let us call them the generalized Slutsky matrix of the first kind. It does not make sense to speak of the symmetry and negative semidefiniteness of this matrix S_{kj} in general.

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However, these properties can still be proved in various special cases, as given in Samuelson [6], Kalman [4, 5], and Chichilinsky and Kalman [1]. These results will be presented in Section 3.

In this paper, we define a matrix K_{ij}, called the generalized Slutsky matrix of the second kind, as $\sum_k(-\partial^2 L/\partial a_i\, \partial x_k)S_{kj}$, a familiar process in tensor analysis or differential geometry. Here $L = f + \sum_s \lambda_s(g_s - b_s)$ denotes the associated Lagrangian function. It is shown in Section 2 that the matrix K_{ij} (representing a covariant two tensor in A) always possesses the symmetric and negative semidefinite properties. The symmetric condition appears here again as an integrability condition, and the negative semidefinite property reflects the maximization nature of this problem. In other words, what we present here is simply a new or intrinsic formulation of the Slutsky matrix in which their basic properties can be stated neatly, in general. Of course, in the cases studied in [1, 4, 5, 6], the symmetric and negative semidefinite properties of the Slutsky matrix of the first kind follow from those of the second kind. These facts are given in Section 3.

In two situations, we can recover f (in special form) from $x(a, b)$, in which (K_{ij}) fulfills the basic properties. These theorems are presented in Sections 4 and 5. In general, the symmetry of K_{ij} does not provide a complete set of integrability coditions, and thus, one cannot recover f (in special form) from $x(a, b)$. An example of this kind is presented in the last section.

Our approach to problems (A) and (B) lies in a systematic use of the differential forms [2, 8]. Therefore, it is more geometric in nature, and formulas involving determinants and cofactors do not appear at all. When dealing with the recovery problem, the Frobenius theorem is needed.

Through this paper, we assume, for simplicity, that all functions considered are smooth (indeed, our arguments work without modification for functions of class C^2 or C^1). We also assume that g is always given and, to each a, the vectors $\partial g_1/\partial x$ $(=(\partial g_1/\partial x_1,..., \partial g_1/\partial x_n)$,..., $\partial g_m/\partial x$ are linearly independent at any point x in X. Thus, for each (a, b) in $A \times B$, the feasible set $\{x \in X \mid g(x, a) = b\}$ is a $(n - m)$ dimensional submanifold if not empty.

We very frequently use the matrix notation to simplify our presentations. Thus, x, a, b, λ, g, dx, da, db, $d\lambda$ stand for column vectors, $\partial L/\partial x$, $\partial g/\partial a$, $\partial x/\partial a$, $\partial x/\partial b$, the appropriate Jacobian matrices, $\partial^2 L/\partial x\, \partial a = (\partial^2 L/\partial x_k\, \partial a_j)$, $\partial^2 L/\partial x^2 = (\partial^2 L/\partial x_k\, \partial x_h)$ the matrix of mixed second order partials, and etc. For a matrix H, ${}^tH$ denotes its transpose.

2. Basic Properties and a Recovery Theorem of x and λ.

The first order consideration for the parameterized constrained problem

$$\max_{x \in X} f(x, a) \qquad \text{subject to } g(x, a) = b, \tag{*}$$

leads to the notion of equilibrium "manifold." In general, the equilibrium manifold $\mathscr{E}^* = \{(x, a, \lambda, b) \mid L_x = 0,\ {}^tL_\lambda = g - b = 0\}$ is a submanifold of dimension $p + m$. We are mainly concerned with a part of $\mathscr{E}^*$ which sits over (a part of) $A \times B$. Thus, our analysis may begin with the following:

DEFINITION 1. Let $x : A \times B \to X, \lambda : A \times B \to R^m$ be two functions. x is said to be an *extreme solution with Lagrangian multiplier* λ for the constrained problem (*) over $A \times B$, iff to each (a, b) in $A \times B$, $(x(a, b), \lambda(a, b))$ is a *critical* point of the Langrangian function $L = f + \sum \lambda (g_s - b_s)$.

Assume (x, λ) is an extreme solution over $A \times B$. Let $\mathscr{E} = \{(x, a, \lambda, b) \in X \times A \times R^m \times B \mid x = x(a, b), \lambda = \lambda(a, b)\}$ be its graph. On

$$\begin{aligned}\mathscr{E}, dL &= \sum_i \frac{\partial L}{\partial a_i} da_i + \sum_s \frac{\partial L}{\partial b_s} db_s \\ &= \sum_i \frac{\partial L}{\partial a_i} da_i + \sum_s (-\lambda_s)\, db_s.\end{aligned}$$

Thus,

$$\begin{aligned}0 = d^2L = \sum_i \Bigg\{ & \sum_k \frac{\partial^2 L}{\partial a_i \partial x_k} \left(\sum_j \frac{\partial x_k}{\partial a_j} da_j + \sum_j \frac{\partial x_k}{\partial b_t} db_t \right) \\ & + \sum_i \frac{\partial^2 L}{\partial a_i \partial a_j} da_j + \sum_s \frac{\partial^2 L}{\partial a_i \partial \lambda_s} \left(\sum_j \frac{\partial \lambda_s}{\partial a_j} da_j + \sum_t \frac{\partial \lambda_s}{\partial b_t} db_t \right) \\ & + \sum_t \frac{\partial^2 L}{\partial a_i \partial b_t} \Bigg\} db_t \wedge da_i \\ & + \left\{ \sum_{i,t} \left(-\frac{\partial \lambda_t}{\partial a_i} \right) da_i \wedge db_t + \sum_{s,t} \left(-\frac{\partial \lambda_s}{\partial b_t} \right) db_t \wedge db_s \right\}.\end{aligned}$$

Here, $\wedge$ means the exterior product between forms. Notice

$$\frac{\partial^2 L}{\partial a_i \partial \lambda_s} = \frac{\partial g_s}{\partial a_i} \qquad \text{and} \qquad \frac{\partial^2 L}{\partial a_i \partial b_t} = 0.$$

Hence, by looking at the terms which involve $da_j \wedge da_i$, $db_t \wedge da_i$, and $db_t \wedge db_s$, respectively, our compatibility conditions read as:

$$\begin{aligned}&\sum_s \frac{\partial \lambda_s}{\partial a_j} \frac{\partial g_s}{\partial a_i} + \sum_k \frac{\partial^2 L}{\partial a_i \partial x_k} \frac{\partial x_k}{\partial a_j} \\ &\qquad = \sum_t \frac{\partial \lambda_t}{\partial a_i} \frac{\partial g_t}{\partial a_j} + \sum_k \frac{\partial^2 L}{\partial a_j \partial x_k} \frac{\partial x_k}{\partial a_i};\end{aligned} \tag{1}$$

$$\frac{\partial\lambda_t}{\partial a_i}+\sum_s \frac{\partial\lambda_s}{\partial b_t}\frac{\partial g_s}{\partial a_i}+\sum_k \frac{\partial^2 L}{\partial a_i\,\partial x_k}\frac{\partial x_k}{\partial b_t}=0; \tag{2}$$

$$\frac{\partial\lambda_s}{\partial b_t}=\frac{\partial\lambda_t}{\partial b_s}. \tag{3}$$

Equation (1) can be replaced by

$$\sum_k \frac{\partial^2 L}{\partial a_i\,\partial x_k}\left(\frac{\partial x_k}{\partial a_j}+\sum_t \frac{\partial g_t}{\partial a_j}\frac{\partial x_k}{\partial b_t}\right)$$
$$=\sum_k \frac{\partial^2 L}{\partial a_j\,\partial x_k}\left(\frac{\partial x_k}{\partial a_i}+\sum_s \frac{\partial g_s}{\partial a_i}\frac{\partial x_k}{\partial b_s}\right), \tag{1'}$$

which is obtained by substituting $\partial\lambda_t/\partial a_i$, $\partial\lambda_s/\partial a_j$, the expressions found in (2) into (1), and using $\partial\lambda_s/\partial b_t=\partial\lambda_t/\partial b_s$.

DEFINITION 2. The matrix $(S_{kj})=(S_{x_k a_j})=(\partial x_k/\partial a_j+\sum_s(\partial x_k/\partial b_s)(\partial g_s/\partial a_j))$ $k=1,\dots,n$, $j=1,\dots,p$, is called a *generalized Slutsky matrix of the first kind*. The matrix $(K_{ij})=(K_{a_i a_j})=(\sum_k-(\partial^2 L/\partial a_i\,\partial x_k)S_{kj})$, $i,j=1,\dots,p$ is called a *generalized Slutsky matrix of the second kind*. In matrix notation, $(S_{kj})=S=\partial x/\partial a+(\partial x/\partial b)(\partial g/\partial a)$, $(K_{ij})=K=-(\partial^2 L/\partial a\,\partial x)S$.

The term S_{kj} can be interpreted as the effect on x_k along a constraint direction of a change in a_j while keeping the value of f constant (i.e., a Slutsky equation). The matrix (K_{ij}) relates to $(\partial^2 L/\partial x_i\,\partial x_j)$ in a very simple way.

PROPOSITION 1. (a) *Set $dx=S\,da$, then, $(\partial g/\partial x)\,dx=0$.*

(b) *${}^t dx(\partial^2 L/\partial x\,\partial x)\,dx={}^t da\,K\,da$, where $dx=S\,da$.*

Proof. (a) Differentiating the equation $g(x,a)=b$, $x=x(a,b)$, one gets $(\partial g/\partial x)\,dx+(\partial g/\partial a)\,da=db$, and $dx=(\partial x/\partial a)\,da+(\partial x/\partial b)\,db$. The result follows by letting $db=(\partial g/\partial a)\,da$.

(b) Differentiating the equation $\partial L/\partial x=0$,

$$\frac{\partial^2 L}{\partial x\,\partial x}dx+\frac{\partial^2 L}{\partial x\,\partial a}da+{}^t\!\left(\frac{\partial g}{\partial x}\right)dx=0.$$

Thus

$${}^t dx\,\frac{\partial^2 L}{\partial x\,\partial x}dx={}^t dx\left[-\frac{\partial^2 L}{\partial x\,\partial a}da-{}^t\!\left(\frac{\partial g}{\partial x}\right)d\lambda\right].$$

Using (a), one gets

$$
\begin{aligned}
{}^t dx \frac{\partial^2 L}{\partial x\,\partial x} dx &= {}^t da\,{}^t S \left(-\frac{\partial^2 L}{\partial x\,\partial a}\right) da \\
&= {}^t da\,{}^t K\, da = {}^t da\, K\, da.
\end{aligned}
$$

Thus, the proof of Proposition 1 is completed.

If $x(a, b)$ is a local maximum for the constrained problem (*) at (a, b), then the ${}^t dx(\partial^2 L/\partial x\,\partial x)dx$ is negative semidefinite on $(\partial g/\partial x)dx = 0$ at (a, b). By Proposition 1, we conclude, in this case, the Slutsky matrix K is semidefinite.

DEFINITION 3. A local maximum $x(a, b)$ with Lagrangian multiplier $\lambda(a, b)$ is said to be *regular* at (a, b) iff ${}^t dx(\partial^2 L/\partial x\,\partial x)\,dx$ is negative *definite* on $(\partial g/\partial x)\,dx = 0$ at (a, b).

Now, we collect the basic properties about (K_{ij}) just obtained, and add a little more.

THEOREM 1. *Suppose x is an extreme solution with Lagrangian multiplier λ over $A \times B$ for the constrained problem (*). Then:*

(1) *the Slutsky matrix K of the second kind has the symmetry property.*

(2) *K is negative semidefinite of rank $\leqslant n - m$ if x is a local maximum.*

(3) *Furthermore, K is of rank $n - m$ iff the local maximum is regular and the map $(a, b) \to x(a, b)$ is a submersion.*

To see statement (3) one needs:

LEMMA 1. *The map $R^p \to \{dx \mid (\partial g/\partial x)dx = 0\}$ defined by $da \to (\partial x/\partial a + (\partial x/\partial b)(\partial g/\partial a))da$ is surjective iff the map $(a, b) \to x(a, b)$ is a submersion.*

Proof. Suppose the map $(a, b) \to x(a, b)$ is surjective. Let dx in R^n be such $(\partial g/\partial x)dx = 0$. Then, there exists da, db such that $dx = (\partial x/\partial a)da + (\partial x/\partial b)db$. Since, $(\partial g/\partial x)dx + (\partial g/\partial a)da - db = 0$. $(\partial g/\partial x)dx = 0$ implies $(\partial g/\partial a)da = db$ or $dx = (\partial x/\partial a + (\partial x/\partial b)(\partial g/\partial a))da$. Conversely, given dx, set $db^* = (\partial g/\partial x)dx$, $dx^* = (\partial x/\partial b)db^*$. For $(\partial g/\partial x)dx^* + (\partial g/\partial a)0 - db^* = 0$, one gets $(\partial g/\partial x)(dx - dx^*) = db^* - db^* = 0$. By hypothesis, there exists da such that $dx - dx^* = (\partial x/\partial a + (\partial x/\partial b)(\partial g/\partial a))da$. Hence, $dx = (\partial x/\partial a)da + (\partial x/\partial b)((\partial g/\partial a)da + db^*)$. Thus, the proof of the lemma is finished.

To show x is a submersion, one often uses:

LEMMA 2. *Suppose x is a regular local maximum. Then,* (a, b) $(a, b) \to x(a, b)$ *is a submersion if (and only if) the map* $da \to (\partial^2 L/\partial x\, \partial a)\, da$ *is surjective modulo* $\partial g_1/\partial x, \ldots, \partial g_m/\partial x$.

Proof. For x is a regular local maximum. The matrix

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{{}^t\partial g}{\partial x} \\[2ex] \dfrac{\partial g}{\partial x} & 0 \end{pmatrix}$$

is nonsingular. Given any dy in R^n, there exists da, db, $d\mu$ such that

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{{}^t\partial g}{\partial x} \\[2ex] \dfrac{\partial g}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} dy \\ d\mu \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial^2 L}{\partial x\, \partial a}\, da \\[2ex] -\dfrac{\partial g}{\partial a}\, da + db \end{pmatrix}.$$

Since $\binom{dx}{d\lambda}$ also satisfies this equation for $\binom{dy}{d\mu}$, by uniqueness of solution, one has $dy = dx = (\partial x/\partial a)\, da + (\partial x/\partial b) db$. Thus, the proof of the lemma is completed.

Remark 1. Suppose the constrained problem (*) has a regular local maximum (x^0, λ^0) at (a^0, b^0). Then

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{{}^t\partial g}{\partial x} \\[2ex] \dfrac{\partial g}{\partial x} & 0 \end{pmatrix}$$

is nonsingular at $(x^0, \lambda^0, a^0, b^0)$. Applying the implicit function theorem to $\partial L/\partial x = 0$, $\partial L/\partial \lambda = 0$, one knows that the problem (*) has a unique regular local maximum solution $x = x(a, b)$, $\lambda = \lambda(a, b)$ near (a^0, b^0) with $x^0 = x(a^0, b^0)$, $\lambda^0 = \lambda(a^0, b^0)$.

Remark 2. Along the equilibrium manifold $\mathscr{E}$, one has

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{{}^t\partial g}{\partial x} \\[2ex] \dfrac{\partial g}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} dx \\ d\lambda \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial^2 L}{\partial x\, \partial a}\, da \\[2ex] \dfrac{\partial g}{\partial x}\, dx \end{pmatrix}.$$

Set

$$\begin{pmatrix} A & {}^tG \\ G & H \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2} & \dfrac{{}^t\partial g}{\partial x} \\ \dfrac{\partial g}{\partial x} & 0 \end{pmatrix}^{-1}$$

if exists. Thus, $dx = -A(\partial^2 L/\partial x\, \partial a)da$ when $(\partial g/\partial x)\, dx = 0$. Consequently, $S = -A(\partial^2 L/\partial x\, \partial a)$ (cf. [1]).

Now, let us present an (easy) recovery theorem for x, λ. Suppose $f(x, a)$ has the form $h(x, a) + \phi(x)$. Then, the compatibility conditions (1)′, (2), and (3), in matrix notation, become:

$$K = {}^tK, \qquad \text{with} \quad K = \frac{\partial^2 h}{\partial a\, \partial x} \left(\frac{\partial x}{\partial a} + \frac{\partial x}{\partial b}\frac{\partial g}{\partial a}\right); \tag{4}$$

$${}^t\!\left(\frac{\partial \lambda}{\partial a}\right) + {}^t\!\left(\frac{\partial g}{\partial a}\right)\left(\frac{\partial \lambda}{\partial b}\right) + \frac{\partial^2 h}{\partial a\, \partial x}\frac{\partial x}{\partial b} = 0; \tag{5}$$

$$\left(\frac{\partial \lambda}{\partial b}\right) = {}^t\!\left(\frac{\partial \lambda}{\partial b}\right). \tag{6}$$

Thus, these conditions are *testable* when one knows h, x, λ, and g.

THEOREM 2. *Assume*: (a) *h, x, λ are given with $g(x(a, b)) = b$ for all a, b and they fulfill the compatibility conditions* (4), (5), *and* (6). (b) *The open sets A, B are simply-connected.* (c) *$x: A \times B \to X$ is a submersion, and $x^{-1}(c)$ is connected for each c in $x(A \times B)$.* (d) *The matrix K is negative semidefinite of rank $n - m$. Then, there exists $\phi: x(A \times B) \to R$, such that, to each (a, b) the object function $f(\cdot, a) = h(\cdot, a) + \phi(\cdot)$ with constraint $g(\cdot, a) = b$ has a regular local maximum at $x(a, b)$ with Lagrangian multiplier $\lambda(a, b)$.*

Proof. Conditions (4), (5), and (6) imply that the form $w = \partial h/\partial a + {}^t\lambda(\partial g/\partial a)\, da + (-{}^t\lambda)\, db$ is closed on $A \times B$ (or $\mathscr{E}$). From (b), $A \times B$ (or $\mathscr{E}$) is simply-connected, thus the closed 1-form w must be exact, say $w = df$, $f = f(a, b)$. Set $\phi(a, b) = f(a, b) - h(x(a, b), a)$. $d\phi = {}^t\lambda((\partial g/\partial a)\, da - db) - (\partial h/\partial x)\, dx = {}^t\lambda(-(\partial g/\partial x)\, dx) - (\partial h/\partial x)\, dx$. Thus, $d\phi = 0$ when $dx = 0$. This fact together with assumption (c) implies that ϕ is constant along $x^{-1}(c)$, or ϕ can be regarded as a function on x. Let ρ be a local right inverse of x. Then, $\phi(x) = \phi \circ \rho(x)$ locally or $\phi = \phi(x)$ is smooth. The established fact $d\phi = {}^t\lambda(-(\partial g/\partial x)\, dx) - (\partial h/\partial x)\, dx$ gives $\partial L/\partial x = \partial h/\partial x + \partial \phi/\partial x + {}^t\lambda(\partial g/\partial x) = 0$. Hence, (x, λ) is an extreme solution. By (c), (d), and Theorem 1, (3), one concludes that x is a regular local maximum. Thus, the proof of Theorem 2 is completed.

3. EXAMPLES

Properties about Slutsky matrix of the first kind can be deduced from that of the second kind, even without knowing λ explicitly.

EXAMPLE 1 (Samuelson [6, pp. 166–168]. Let $f = U(x)$ be a utility function, and $P^\alpha \cdot x = I^\alpha$, $\alpha = 0,\dots, r$ be $(r+1)$ constraints, where $\{P^\alpha = {}^t(p_1^\alpha,\dots, p_n^\alpha)\}$ are linearly independent. Assume $x = x(p, I)$ is a regular local maximum with Lagrangian multiplier $\lambda \ll 0$ (i.e., $\lambda_0 < 0,\dots, \lambda_r < 0$) over some open set in (p, I) space. Set $S^\alpha = (\partial x_i/\partial p_j^\alpha + x_j(\partial x_i/\partial p_j^\alpha))$ a substitution matrix for each α. Then, (a) the matrix S^α is symmetric, negative semidefinite and of rank $n-(r+1)$. (b) S^α's are propositional to each other.

To verify these properties about S^α, one can proceed as follows. For $\partial g_s/\partial p_j^\beta = \delta_\beta^s x_j$ ($\delta_\beta^s = 0$ if $s \neq \beta$, $\delta_\beta^s = 1$, if $s = \beta$), $S_{x_k p_j^\beta} = \partial x_k/\partial p_j^\beta + (\partial x_k/\partial b_j)\, x_j$. For $\partial^2 L/\partial p_i^\alpha \partial x_k = \lambda_\alpha \delta_i^k$, $K_{p_i^\alpha p_j^\beta} = -\lambda_\alpha S_{x_i p_j^\beta}$ or $K = (K_{\alpha\beta}) = (-\lambda_\alpha S^\beta)$. By Theorem 1, (1), $-\lambda_\alpha S_{x_i p_j^\beta} = -\lambda_\beta S_{x_j p_i^\alpha}$. Set $\alpha = \beta$, one gets $S_{x_i p_j^\alpha} = S_{x_j p_i^\alpha}$, or $S^\alpha = {}^t(S^\alpha)$. Hence, $-\lambda_\alpha S^\beta = -\lambda_\beta S^\alpha$ or $S^\alpha/\lambda_\alpha = S^\beta/\lambda_\beta = S^0/\lambda_0$. Thus, we get (a), (b) except that S^α is negative semidefinite of rank $n-(r+1)$,

$$
\begin{aligned}
{}^t dp\, K\, dp &= \sum_{\alpha,\beta,i,j} dp_i^\alpha K_{p_i^\alpha p_j^\beta}\, dp_j^\beta = \sum -\frac{\lambda_\alpha \lambda_\beta}{\lambda_0} S_{ij}^0\, dp_i^\alpha\, dp_j^\beta \\
&= \sum_{i,j} -\frac{S_{ij}^0}{\lambda_0} \left(\sum_\alpha \lambda_\alpha\, dp_i^\alpha\right)\left(\sum_\beta \lambda_\beta\, dp_j^\beta\right).
\end{aligned}
$$

By Lemma 2, Theorem 1, (3) ${}^t dp\, K\, dp$ is negative semidefinite of rank $n-(r+1)$. From the above equation, one concludes S^0 or S^α enjoys the same properties.

EXAMPLE 2 (Kalman [4]). Consider a utility function $u(q, p)$ in the form $f(q_1 + (1/2)p_1^2,\dots, q_n + (1/2)p_n^2) + h(p_1,\dots, p_n)$ with a budget constraint $p \cdot q = y$. Suppose, to each (p, y) in some open set in (p, y) space, it has a regular local maximum $q = q(p, y)$ with Lagrangian multiplier $\lambda(p, y) < 0$. Set $S = (\partial q_k/\partial p_j + (\partial q_k/\partial y)(q_j - p_j^2))$. Then, S is a symmetric, negative semidefinite matrix of rank $n-1$. To justify these properties, it suffices to observe that $K = (-\lambda)S$, K, S the Slutsky matrices of first and second kind for our constrained problem in Q coordinates $(Q_k = q_k + (1/2)p_k^2)$.

Making a p-depedent change of coordinates $Q_k = q_k + (1/2\, p_k^2$, to each (p, y), $Q(p, y) = (Q_k(p, y)) = (q_k(p, y) + (1/2)p_k^2)$ is a regular local maximum with Lagrangian multiplier $\lambda(p, y)$ for $u = f(Q) + h(p)$, with constraint $g = \sum_j p_j(Q_j - (1/2)p_j^2) = y$. For $\partial Q_k/\partial p_j = \partial q_k/\partial p_j + \delta_{kj} p_k$, $\partial Q_k/\partial y = \partial q_k/\partial y$, $\partial^2 L/\partial Q_k \partial p_i = \lambda \delta_{ki}$ and $\partial g/\partial p_j = Q_j - (3/2)p_j^2$, one has

$$S_{kj} = \frac{\partial Q_k}{\partial p_j} + \frac{\partial Q_k \partial g}{\partial y\, \partial p_j} = \frac{\partial q_k}{\partial p_j} + \frac{\partial q_k}{\partial y}\left(Q_j - \frac{3}{2} p_j^2\right)$$
$$= \frac{\partial q_k}{\partial p_j} + \frac{\partial q_k}{\partial y}(q_j - p_j^2),$$

and

$$K_{ij} = \sum_k - \frac{\partial^2 L}{\partial p_i\, \partial Q_k} S_{kj} = (-\lambda) S_{ij}.$$

Remark 3. The equation at line 18, p. 508 in [4], should read:

$$\sum_{j=1} \frac{D_{ji}}{D} u_{jh} p_h = \begin{cases} -\dfrac{\partial q_i}{\partial y} p_h^2, & h \neq i, \\[2ex] -\dfrac{\partial q_j}{\partial y} p_i^2 + p_i, & h = i. \end{cases}$$

After making such a change, our result is consistent with that in [4].

EXAMPLE 3 (Kalman, Dusansky, and Wickstrom [5]). Let $u = u(q, M, m, p)$ be an object function in the form $u = c_1 p \cdot q + c_2 \hat{p} \cdot q + \zeta(q, M, m) + \xi(p)$, and $p \cdot q + M + m = b$, $\hat{p} \cdot q + m = c$ be two constraints. Here, $(p, b, c) \in R^n \times R \times R$ stands for the parameters, ${}^t x = (q, M, m) \in R^n \times R \times R$ the variables, c_1, c_2 constants ζ, ξ functions, and $\hat{p} = {}^t(p_1, ..., p_l, 0 \cdots 0) \in R^n$ for some fixed l. Simple computations give the formula of the Slutsky matrix of the first kind

$$S_{q_k p_j} = \begin{cases} \dfrac{\partial q_k}{\partial p_j} + \dfrac{\partial q_k}{\partial b} q_j + \dfrac{\partial q_k}{\partial c} q_j, & j \leqslant l \\[2ex] \dfrac{\partial q_k}{\partial p_j} + \dfrac{\partial q_k}{\partial b} q_j & j > l, \text{ and etc.} \end{cases}$$

Partition S as

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \end{pmatrix}$$

where S_{11}, S_{22} are matrices of order $l \times l$, $(n-l) \times (-l)$, respectively. Now,

$$\left(\frac{\partial^2 L}{\partial p_i\, \partial x_k}\right) = \begin{pmatrix} (c_1 + c_2 + \lambda + \mu) I_{l \times l} & 0 & 0 \\ 0 & (c_1 + \lambda) I_{(n-l) \times (n-l)} & 0 \end{pmatrix}.$$

Therefore,

$$K = -\begin{pmatrix} (c_1 + c_2 + \lambda + \mu)S_{11} & (c_1 + c_2 + \lambda + \mu)S_{12} \\ (c_1 + \lambda)S_{21} & (c_1 + \lambda)S_{22} \end{pmatrix}.$$

Consequently, by Theorem 1, if $c_1 + c_2 + \lambda + \mu < 0$ and $c_1 + \lambda < 0$, then S_{11}, S_{22} are symmetric and negative semidefinite matrices.

EXAMPLE 4 (Chichilnisky and Kalman [1]). Consider an object function $f = \gamma a \cdot x + f^1(x) + f^2(a)$ with contraints $g_s = \delta_s a \cdot x + g_s^1(x) + g_s^2(a) = b_s$, $s = 1,..., m$. Here, $(a, b) \in R^n \times R^m$ denotes the parameters and $x \in R^n$ the variables. γ, δ_s constants and f^1, f^2, g_s^1, g_s^2 arbitrary functions. Of course we assume $\partial g_1/\partial x,..., \partial g_m/\partial x$ are linearly independent. For a regular local maximum $x = x(a, b)$, its Slutsky matrices are, by simple computations,

$$(S_{kj}) = \left(\frac{\partial x_k}{\partial a_j} + \sum_s \frac{\partial x_k}{\partial b_s}\left(\delta_{sj}\delta_s + \frac{\partial g_s^2}{\partial a_j}\right)\right),$$

$$(K_{ij}) = \left(-\gamma + \sum_s \lambda_s \delta_s\right)(S_{ij}).$$

Therefore, by Theorem 1, (3) and Lemma 2, (S_{ij}) is a symmetric negative semidefinite matrix of rank $n - m$, when $\gamma + \sum_s \lambda_s \delta_s < 0$.

4. A RECOVERY THEOREM FOR A SINGLE CONSTRAINT

If only the optimal solution $x = x(a, b)$ but not the multiplier $\lambda(a, b)$ is known, then the recovery problem in general bears no interest. For one can always recover an object function in the general situation. Indeed, by adding $g_{m+1}(x, a),..., g_n(x, a)$ to $g_1(x, a),..., g_m(x, a)$ so that to each a, $g_1,..., g_n$ defines a coordinate system near $x(a, b)$, the desired objective function $f(x, a)$ can be taken as $f(x, a) = -\sum_{j=m+1}^{n} (g_j(x, a) - g_j(x(a, g_1(x, a),..., g_m(x, a)), a)^2$. Therefore, the interesting recovery problem, without knowing λ, is for those $f(x, a)$ in a certain special form. In this section, we deal with such a recovery theorem, both in local and global versions, with a single constraint.

Consider a parameterized constrained problem with an object function $f(x, a) = \phi(x)$ having no critical points, and a constraint given by a scalar function $g(x, a) = b$. Assume $x = x(a, b)$ is a regular local maximum with Lagrangian multiplier $\lambda(a, b)(<0)$. Hence,

$$(\tilde{K}_{ij}) = \frac{K_{ij}}{-\lambda} = \left(\sum_k \frac{\partial^2 g}{\partial x_k \partial a_i}\left(\frac{\partial x_k}{\partial a_j} + \frac{\partial x_k}{\partial b}\frac{\partial g}{\partial a_j}\right)\right)$$

defines a symmetric, negative semidefinite matrix of rank $\leqslant n-1$, a condition testable when x, g are known but not λ.

Conversely, we have

THEOREM 3 (Local version). *Let $g(x,a)$ be a scalar function with $(\partial g/\partial x)(x^0, a^0) \neq 0$, and $x = x(a,b)$ a submersion defined near (a^0, b^0) with $x^0 = x(a^0, b^0)$, $g(x(a,b), a) = b$. Suppose (K_{ij}) is symmetric, negative semidefinite of rank $n-1$. Then, there exists $\phi(x)$ defined near x^0, such that to each (a,b) near (a^0, b^0), $x(a,b)$ maximizes ϕ subject to $g(x,a) = b$.*

To obtain this theorem, we use the Frobenius theorem in the form (see [8]): Let ω be a 1-form with $d\omega \neq 0$ near a point p in R^l. There exists a function of ϕ such that $d\phi = \lambda\omega$ for some $\lambda \neq 0$ near p iff $d\omega = \mu \wedge \omega$ for some 1-form μ.

Proof. Set

$$\omega = \sum_i \frac{\partial g}{\partial a_i} da_i - db \cdot d\omega$$

$$= \sum_i \left[\sum_j \frac{\partial^2 g}{\partial a_j \partial x_k} \frac{\partial x_k}{\partial a_j} da_j + \sum_j \frac{\partial^2 g \, da_j}{\partial a_i \partial a_j} + \sum_k \frac{\partial^2 g}{\partial a_i \partial x_k} \frac{\partial x_k}{\partial b} db \right] da_i.$$

Thus, $d\omega = \mu \wedge \omega$, with

$$\mu = \sum_i \left(\sum_k \frac{\partial^2 g}{\partial a_i \partial x_k} \frac{\partial x_k}{\partial b} \right) da_i.$$

Here, we need $\tilde{K}_{ij} = \tilde{K}_{ji}$. By the Frobenius theorem, there exists ϕ defined near (a^0, b^0) such that $d\phi = \lambda\omega$ with $\lambda < 0$. For $d\phi = \lambda\omega = -\lambda \sum_k (\partial g/\partial x_k) \, dx_k)$, $dx = 0$ implies $d\phi = 0$. This fact, together with that $x = x(a,b)$ is a submersion, guarantees that ϕ can be regarded as a function of x and $x(a,b)$ an extreme solution. By Proposition 1, and Theorem 1, (3), we conclude that (x) has the desired properties and the proof of Theorem 3 is completed.

To obtain a global version of Theorem 3, one needs global result of the Frobenius type theorem.

THEOREM (Thomas [3]). *Let $\omega = (\sum_i \xi_i \, da_i) - db$ be a 1-form defined over a simply-connected domain $\Omega \times R$. Suppose*

(a) $$\frac{\partial \xi_i}{\partial a_j} + \frac{\partial \xi_i}{\partial b} \xi_j = \frac{\partial \xi_j}{\partial a_i} + \frac{\partial \xi_j}{\partial b} \xi_i \qquad \text{for all } i, j.$$

(b) $|\partial\xi_i/\partial b|$ *is bounded over sets in the form* $C \times R$, *where* $C =$ *compact set in* Ω.

Then, there exists a function $\phi(a, b)$ *such that* $d\phi = \lambda\omega$ *for some* $\lambda < 0$ *over* $\Omega \times R$.

Now, we can obtain

THEOREM 3′ (Global version). *Let* $g(x, a)$ *be a scalar function with* $(\partial g/\partial x)(x, a) \neq 0$, *and* $x = x(a, b)$ *a submersion defined over a simply-connected domain* $\Omega \times R$ *with* $g(x(a, b), a) = b$. *Suppose that*:

(a) $(\tilde{K}_{ij})$ *is a symmetric, negative semidefinite matrix of rank* $n-1$.

(b) $|\sum_k(\partial^2 g/\partial a_l\, \partial x_k)(\partial x_k/\partial b)| \leqslant K_c$ *over sets in the form* $C \times R$, *where* $C =$ *compact set in* Ω, K_C *a constant depending on* C.

(c) $x^{-1}(c)$ *is connected, for each* c *in* $x(\Omega \times R)$. *Then, there exists a function* $\phi(x)$ *defined on* $x(\Omega \times R)$ *such that to each* (a, b), $x(a, b)$ *is a regular local maximum of* $\phi(x)$ *subject to* $g(x, a) = b$.

The proof of this theorem, is very similar to that of Theorem 3. Essentially, one need only replace the Frobenius theorem by Thoma's theorem described above. We leave the details to the readers.

Remark 4. I do not know how to weaken the assumption (b) so that one can still obtain the same conclusion.

5. A RECOVERY THEOREM IN THE CASE $g = g(x)$ AND A NONRECOVERABLE EXAMPLE FROM ITS SLUTSKY MATRIX

(A) Let us take the object function in the form $f(x, a) = h(x, a) + \phi(x)$, and vector constraint $g = g(x) = b$. The Slutsky matrix

$$(K_{ij}) = \left(\sum_k \frac{\partial^2 L}{\partial x_k\, \partial a_j} \left(\frac{\partial x_k}{\partial a_j} + \sum_s \frac{\partial x_k}{\partial b_s}\frac{\partial g_s}{\partial a_j}\right)\right)$$
$$= \left(\sum_k \frac{\partial^2 h}{\partial x_k\, \partial a_j}\, \frac{\partial x_k}{\partial a_j}\right)$$

has, of course, symmetric and negative semidefinite properties, which can be tested when x, h, g are given. Indeed, we have a recovery theorem in this situation.

THEOREM 4. *Let* $x = x(a, b)$ *be a submersion defined over some simply connected open set* $A \times B$ *in* $R^p \times R^m$ *with* $g(x(a, b)) = b$. *Suppose* (a) (K_{ij}) *is* *a symmetric, negative semidefinite matrix of rank* $n-m$. (b) $x^{-1}(c)$ *is connected for each* c *in* $x(A \times B)$. *Then, there exists a function* $\phi(x)$ *such that, to each* (a, b), $x(a, b)$ *is a regular local maximum for the object function* $f(x, a) = h(x, a) + \phi(x)$ *subject to* $g(x) = b$.

Proof. Let $\mu = \mu(a, b)$ solve the equation $\partial\mu/\partial a_i + \partial h/\partial a_i\ (x(a, b), a) = 0$, for $i = 1,\dots, p$. This is possible by the compatibility condition $K_{ij} = K_{ji}$. Set $\lambda_s = \partial\mu/\partial b_s$. By Theorem 2 in Section 1, it suffices to check conditions (4), (5), and (6). Conditions (4) and (6) are clearly valid. The verification of condition (5) follows from direct computation:

$$\frac{\partial}{\partial a_i}\left(\frac{\partial\mu}{\partial b_t}\right) + \sum_k \frac{\partial^2 h}{\partial a_i \partial x_k}\left(\frac{\partial x_k}{\partial b_t}\right)$$
$$= \frac{\partial\mu}{\partial b_t}\left[\frac{\partial\mu}{\partial a_i} + \frac{\partial h}{\partial a_i}(x(a, b, a))\right] = 0.$$

(B) First, let us derive a (complete) set of integrability conditions for the following constrained optimization problem with parameters $p > 0$, $q > 0$.

(**) Maximize $U(x, y, z)$ subject to $x + py = b$ and $qy + z = c$. Let $x = x(p, q, b, c)$ be a regular local maximum solution with Lagrangian multiplier (λ_1, λ_2), $\lambda_1 < 0$. For $dU = dL = \sum_i(\partial L/\partial a_i)\, da_i + \sum_s(\partial L/\partial b_s)\, db_s$ on the equilibrium manifold, one has $dU = \lambda_1(y\, dp - db) + \lambda_2(y\, dq - dc)$. Since $\partial U/\partial b = -\lambda_1 \neq 0$; by implicit function theorem, the equation $U(x(p, q, b, c)\cdot, \cdot\,) = \text{constant}$ can be solved as $b = b(p, q, c)$ and

$$\frac{\partial b}{\partial p} = y,$$

$$\frac{\partial b}{\partial q} = -\mu y,$$

$$\frac{\partial b}{\partial c} = \mu, \qquad \text{with} \quad \mu = \lambda_2/-\lambda_1.$$

Hence, a (complete) set of compatibility conditions are

$$\frac{\partial y}{\partial c} + \frac{\partial y}{\partial b}\mu = \frac{\partial\mu}{\partial p} + \frac{\partial\mu}{\partial b}y, \tag{7}$$

$$\frac{\partial y}{\partial q} + \frac{\partial y}{\partial b}(-\mu y) = -\mu\left[\frac{\partial y}{\partial p} + \frac{\partial y}{\partial b}\right]y \tag{8}$$
$$-y\left[\frac{\partial\mu}{\partial p} + \frac{\partial\mu}{\partial b}\right]y,$$

and

$$\frac{\partial \mu}{\partial q} + \frac{\partial \mu}{\partial b}(-\mu y) = -\mu\left[\frac{\partial y}{\partial c} + \frac{\partial y}{\partial b}\mu\right] - y\left[\frac{\partial \mu}{\partial c} + \frac{\partial \mu}{\partial b}\mu\right]. \tag{9}$$

PROPOSITION 2. *Set* $y = p + q^2/2$, $x = b - py$, $z = c - qy$, *with* $p > 0$, $q > 0$. *Then,* (a) *the Slutsky matrix* K *is symmetric iff* $\lambda_1 = \lambda_2 = 0$ *or* $\mu = \lambda_2/-\lambda_1 = -q$. *It is also negative semidefinite of rank* 1 *if* $\mu = -q$ *and* $\lambda_1 < 0$.

(b) *There does not exist an object function* $U = U(x, y, z)$ *which gives* $x = b - py$, $y = p + q^2/2$, $z = c - qy$ *as a regular local maximum subject to* $x + py = b$, $qy + z = c$.

Proof. (a) Clearly,

$$K = \begin{pmatrix} \lambda_1\left(\frac{\partial y}{\partial p} + y\frac{\partial y}{\partial b}\right) & \lambda_1\left(\frac{\partial y}{\partial q} + y\frac{\partial y}{\partial c}\right) \\ \lambda_2\left(\frac{\partial y}{\partial p} + y\frac{\partial y}{\partial b}\right) & \lambda_2\left(\frac{\partial y}{\partial q} + y\frac{\partial y}{\partial c}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & \lambda_1 q \\ \lambda_2 & \lambda_2 q \end{pmatrix}.$$

Hence, $K = {}^tK$ iff $\lambda_2 = \lambda_1 q$. When $\mu = -q$, $\lambda_1 < 0$, $K = (-\lambda_1)\begin{pmatrix} -1 & -q \\ -q & -q^2 \end{pmatrix}$ and thus negative semidefinite of rank 1.

(b) Assume on the contrary, we have $\lambda_1 = \lambda_2 = 0$ or $\mu = \lambda_2/-\lambda_1 = -q$, $\lambda_1 < 0$. If $\lambda_1 = \lambda_2 = 0$ at some point, then U has a critical point and x, y, z become constants which is impossible. In the case $\lambda_1 < 0$, and $\mu = -q$. It should fulfill conditions (7), (8), and (9). Indeed, conditions (7), (8) are valid but not (9). Consequently, no such object function $U(x, y, z)$ exist.

Thus, combining parts (a) and (b), the proof of Proposition 2 is completed.

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